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# Meanders: A personal perspective to the memory of Pierre Rosenstiehl

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## ABSTRACT

J'errais dans un méandre ;  
J'avais trop de partis,  
Trop compliqués, à prendre...  
(Edmond Rostand,  
Cyrano de Bergerac)

Meander is a self-avoiding closed curve on a plane which intersects a straight line in a given set of points. Meander is a very simple object. In the elementary school, we may ask children to draw a few meanders and to admire their strange beauty. In the middle school, we may ask children to perform an exhaustive search of the meanders with a small number of intersections with the line. Then, gradually, we start to perceive an incredible profoundness of the subject, whose relations go from enumeration to quantum field theory and string theory. Pierre Rosenstiehl was one of the pioneers in the study of the algorithmic aspects of meanders, and he also was a passionate connoisseur of labyrinths, of which the meanders are a particular case.

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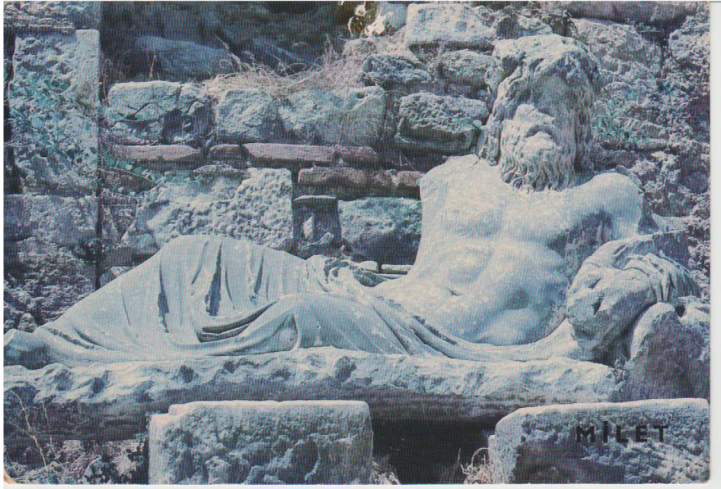
## 1. Introduction

This paper is a follow-up of my talk, in the presence of Pierre Rosenstiehl, at the conference in his honor held at the University of Bordeaux in April 2015 entitled *Labyrinth day*.

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**Fig. 1.** The god Meandros, one of the sons of Oceanus and Tethys (Milet, Asia Minor). A copy of a postcard sent to the author by Jean-Guy Penaud, August 1992.

The literature on labyrinths is immense. For my bibliography, I chose only two books. The first is *The Maze and the Warrior: Symbols in Architecture, Theology, and Music*, by Craig Wright [33], because of a wide range of subjects treated, from symbolic of an epic journey through this sinful world to salvation in religious books, and to certain compositions by J. S. Bach. The second one is *Le labyrinthe des jours ordinaires*, by Pierre Rosenstiehl [26]: he offered me a copy during the above-mentioned *Labyrinth day*.

*Meanders* represent a subclass of labyrinths: properties which distinguish them among all labyrinths are explained in detail in the preprint of Phillips [23] (see also Phillips's web-site [22]). Mainly, meanders are *unicursal* labyrinths, that is, without bifurcations or branching: at no place there is a choice as to what direction to take. The path just goes back and forth. *A priori* it is difficult to believe that something interesting may come out of these simple objects. And yet...

There are quite a few examples in Mathematics when the first impetus to study certain objects comes from their visual beauty, and only later, gradually, it becomes clear that there is a profound mathematical structure behind them. Take, for example, complex dynamical systems: after the advent of computers the striking beauty of fractals inspired a great enthusiasm of many researchers. Meanders represent a similar, though more modest, example of this phenomenon. The term *meander* was coined by Vladimir Arnold. Maybe it is worth saying that historically Meander was the name of a river in the Asia Minor, in the ancient region Caria.

The Maeander<sup>1</sup> was so celebrated in antiquity for its numerous windings, that its classical name "Maeander" became, and still is, proverbial.

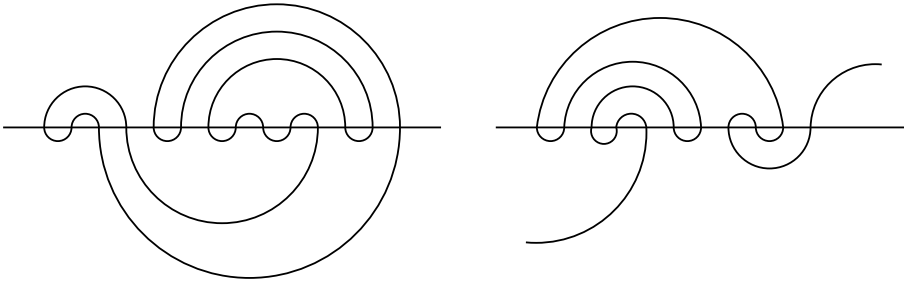
(Wikipedia)

Today this territory belongs to Turkey, and the river is called Büyük Menderes (which means Grand Menderes) (see Fig. 1).

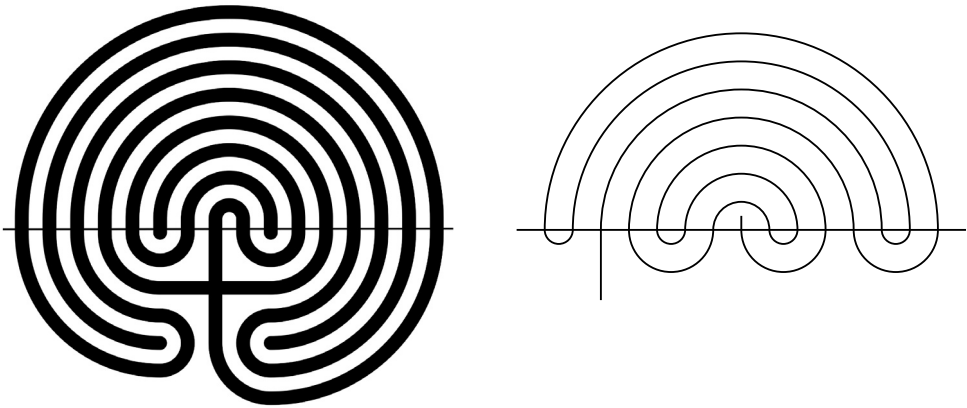
In order to continue our discussion we need to know what a meander is.

**Definition 1.1** (*Meander*). A *meander* of order  $n$  is a self-avoiding plane closed curve which intersects a straight line in  $2n$  points. Meander is defined up to an isotopy of the plane which preserves the intersection points.

<sup>1</sup> Maeander is a Latin denomination, while the Ancient Greek denomination is Μαίανδρος (Μαίανδρος).



**Fig. 2.** Left: a closed meander. There are 14 intersection points, hence this meander is of order 7. Right: an open meander with 11 intersection points.



**Fig. 3.** Left: the classical image of the Cretan labyrinth. It is engraved on many ancient coins, medallions, etc. It may also be found in the medieval Hebrew book *Sefer Haftarot* (the Book of Excerpts), where it represents the seven walls of Jericho. We have added to this image a horizontal line in order to relate the passage through the labyrinth to the meander shown on the right. Right: a half-open meander corresponding to the passage from the entry of the Cretan labyrinth to its center.

An example of a meander is shown in Fig. 2, left.

The above definition admits several variations:

- We may replace the plane by the sphere, considering the straight line as its equator. Thus, a *spherical meander* may be considered as a pair of non-self-intersecting circles which intersect each other in  $2n$  points. In order to be coherent with Definition 1.1, we mark one of the intersection points.
- *Open meanders*: they are not circles but lines coming from  $-\infty$  and going to  $+\infty$ . They may intersect the straight line in even number of points and return to the same half-plane, or intersect it in odd number of points and leave the picture in the other half-plane.
- *Half-open meanders*: they come from infinity but get stuck somewhere in a dead-end. In labyrinths, such a dead-end is often a center, a goal to reach.

*Closed meanders* are those which correspond to Definition 1.1. The adjective “closed” is used when it is necessary to oppose them to other species of meanders described above. We will mainly work with closed meanders.

Fig. 3 illustrates the relation between meanders and labyrinths: a meander is a path inside a labyrinth without bifurcation. We may say that meander is a *dual* to labyrinth. In Fig. 4, we



**Fig. 4.** Left: Theseus kills Minotaur. Detail of a panel (1510–1515) at the Petit Palais d'Avignon, by an anonymous painter traditionally called Maestro dei Cassoni Campana. Right: a schematic representation of the same labyrinth. We see that the labyrinth is *unicursal*: one does not need an Ariadne's thread in order to find a way out. One just has to keep going, and to believe that his efforts will ultimately lead him to an exit.

see a more sophisticated example of a “simple alternating transit maze”, as they were called by Phillips [23]. I did not try to draw the corresponding meander.

## 2. A few historical remarks

A mere two decades ago, it was quite a task to find bibliographical sources for one's work. Today, the difficulty is opposite: one has to find way in the labyrinth of references to references to references, and this labyrinth is in no way unicursal. Fortunately, the study of meanders or, more exactly, of structures similar to them, was not very active. Here are a few references I was able to find (and to choose) concerning the prehistory of our subject.

- A combinatorial problem similar to that of meanders is that of the folding of a strip of stamps. Apparently, it was first stated on page 29 of a little book [28] (1926) by André Sainte-Laguë. Certain authors call this book “the zeroth book on graph theory”: indeed, the book *Theory of Finite and Infinite Graphs* by Dénes König, which is traditionally considered as the first book on the subject, was published in 1936, thus, ten years later.
- The stamp-folding problem was attacked by Jacques Touchard in 1950 [30] (see Fig. 5), and in 1968 there appeared two papers treating this problem: one by John Koehler [16], and one by W. F. Lunnon [21]. The problem was even mentioned in the *New York Times* Science section for January 27, 1987 (see [23]).
- The next step was taken by Pierre Rosenstiehl and his various co-authors, see [25,27] in 1984, and [11] in 1986. They called the objects under study *Jordan permutations* (see Fig. 6) and considered various algorithmic problems related to them. By the way, the fact that these permutations may be sorted in linear time implies the exponential (and not factorial) upper bound for their number, a question which will be raised several times in the future.
- Certain sources remain unavailable. For example, according to Arnold [3], in 1978 Yakov Eliashberg proved a generalization of the theorem of Poincaré–Birkhoff (see below) to surfaces

taux que pose la théorie des permutations. Dans le graphique ( $\Gamma$ ) nous supposerons toujours que la permutation est bonne; on n'aura donc pas de point

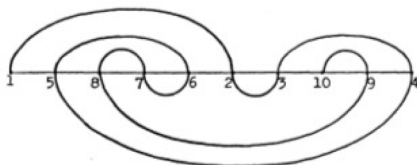


FIGURE 1

double de 1<sup>re</sup> espèce, mais on peut avoir 0, 1, 2, ... p.d.f. et, si  $T_k^n$  est le nombre des figures qui présentent  $k$  p.d.f., on aura

$$P_n = T_0^n + T_1^n + T_2^n + \dots$$

Fig. 5. An excerpt from Jacques Touchard's paper [30], 1950. Sorry for the quality of the image.

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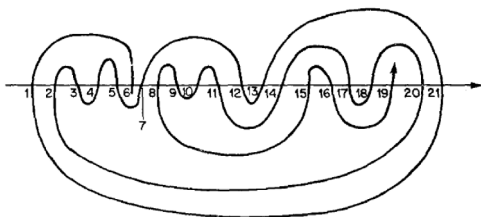


FIG. 1. A Jordan curve corresponding to the sequence 6, 1, 21, 13, 12, 7, 5, 4, 3, 2, 20, 18, 17, 14, 11, 10, 9, 8, 15, 16, 19.

tions in the order they occur along the polygon in linear time by computing the intersection of the line with each side of the polygon. (We

Fig. 6. An excerpt from Pierre Rosenstiehl's paper [25], 1984.

of arbitrary genus, and he used meanders in his proof. Unfortunately, this proof was published as a preprint of the university of Syktyvkar<sup>2</sup> [9].

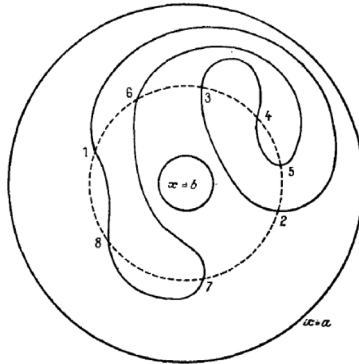
A parallel story took place in Moscow, and its central figure was Vladimir Igorevich Arnold (1937–2010). Arnold was a man of very strong convictions, not always conventional. For him, the greatest mathematicians of all times were Newton and Poincaré. Such people as Gauss or Euler were dwarfs as compared to these two giants. Naturally, Arnold became the chief editor of the Russian translation of the four-volume edition of Collected Works of Henri Poincaré. The very last paper, published in 1912, the year of Poincaré's death, was called "Sur un théorème de géométrie". Poincaré did not succeed to prove the theorem in question but he still decided to share his ideas with the mathematical community. (The theorem was proved the next year by George Birkhoff.)

An image in this paper (see Fig. 7) apparently impressed Arnold. Traditionally, he devoted the first session of his seminar to proposing a list of problems to work on. So, in its turn, he gave a problem of enumeration of meanders (the term is also his). He also wrote an article about them for *Kvant*, a scientific-popular magazine for high-school students, and used them as a tool in [2].

There was also another part to this story. Arnold, while being one of the greatest mathematicians of his generation, rather late, near fifty, got acquainted with the Catalan numbers. He was full of

<sup>2</sup> Syktyvkar is a town 1000 km to north-east from Moscow. The average yearly temperature over there is 1.3 °C and may descend to -46.6 °C in January, and to -5 °C in June. In 1989, professor Eliashberg moved from Syktyvkar to Stanford where the climate is more clement.

extrêmes sont consécutifs, ainsi que leurs numéros. Ce qui caractérise la distribution normale, c'est que les rangs se suivent dans le même ordre que les numéros, et que l'on peut toujours s'arranger pour que le rang d'un point soit égal à son numéro.



(Fig. 1).

On se rendra mieux compte de ce qui précède en se reportant à la figure 1; sur cette figure on s'est servi de l'image circulaire; les deux circonférences extrêmes  $x = a$  et  $x = b$  sont représentées en trait plein; il en est de même de la courbe  $X = c$ ; quant à la circonférence  $x = c$ , elle est en trait pointillé; on voit que les arcs élémentaires extérieurs sont 12, 34, 56 et 78, et les arcs élémentaires intérieurs 23, 45, 67

Fig. 7. Excerpt from Henri Poincaré's paper [24], 1912.

enthusiasm and talked about them to everybody who was ready to listen. One can imagine his reaction when he learned that the meander problem is related to these numbers. (By the way, when in 1990, already in France, I talked about this problem to Philippe Flajolet (1948–2011), his first reaction was that it should be related to the Catalan numbers.)

All these activities resulted in two papers [17,18] by Sergei Lando, who was a former Arnold's student, and me.

Here begins my own meandering way from Moscow to Bordeaux. Many people played an important role in changing the course of my life, but I would mention here only one of them: Michel Mendès France (1936–2018). During all his career as a mathematician Michel was taken up with curly curves, as is shown by his paper [8] (and some unpublished preprints), his graphical works, and also his lectures to students when he explained how to measure the length of such curve by counting the numbers of intersection points of the curve with a randomly laid ruler. So it was a great chance for me when Michel was appointed as a referee for the talk I proposed to the FPSAC conference held in Bordeaux in April 1991.

Well, after this long introduction, let us move to Mathematics.

### 3. Preliminaries

**Definition 3.1 (Arc Diagram).** An arc diagram of order  $n$  is a collection of  $n$  non-intersecting arcs, each of them connecting two points of a set of  $2n$  points on a straight line.

Fig. 12 shows all the five arc diagrams of order 3.

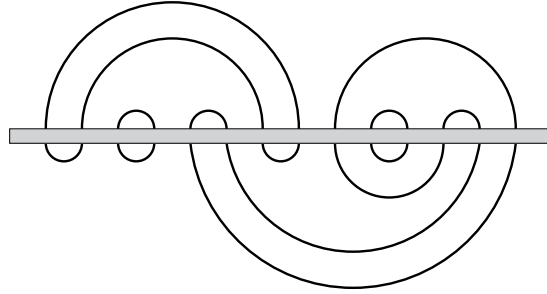
It is well-known that the number of the arc diagrams of order  $n$  is the  $n$ th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

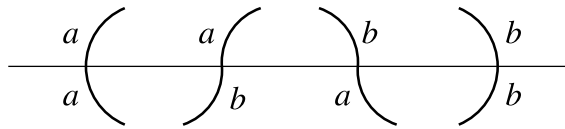
Asymptotic for these numbers easily follows from the Stirling formula:

$$C_n \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2},$$





**Fig. 8.** If we glue the upper and the lower arc diagrams to each other we get a meander system. In this example the system is composed of four components.



**Fig. 9.** An alphabet to encode meander systems:  $aa = l$  (left),  $ab = u$  (up),  $ba = d$  (down),  $bb = r$  (right).

and their generating function is

$$\sum_{n=0}^{\infty} C_n t^n = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

Arc diagrams of order  $n$  may be encoded by the Dyck words of length  $2n$ . Dyck words are the words  $w$  in the alphabet  $\{a, b\}$  such that  $|w|_a = |w|_b$ , and for any prefix  $u$  of  $w$  we have  $|u|_a \geq |u|_b$ . Each letter  $a$  corresponds to the left end of an arc, and a letter  $b$  corresponds to the right end of an arc.

**Definition 3.2 (Meander System).** Let us take two arc diagrams of the same order  $n$ , turn one of them upside-down, and glue them together along their horizontal lines, see Fig. 8. What we get is a meander system which may be composed of several meanders, their number varying from one to  $n$ . Thus, a meander is a meander system with a single component.

**Notation 3.3 (Number of Meanders).** We denote by  $M_n$  the number of meanders of order  $n$ .

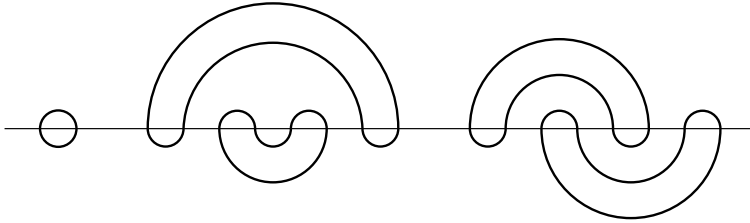
It is easy to verify that any upper arc diagram may be completed to a meander (that is, a meander system with a single component) by an appropriate lower arc diagram. Thus, we obtain trivial upper and lower bounds:

$$C_n \leq M_n \leq C_n^2.$$

We may use an alphabet of four letters  $\{l, u, d, r\}$  in order to encode the meander systems: the meaning of the letters is clear from Fig. 9.

Note however that, while the language of the Dyck words admits a simple characterization by a context-free grammar, and hence its generating function is algebraic, the language of the meander systems does not have such a representation. Indeed, the generating function for the squares of Catalan numbers is not algebraic: it may be represented by an elliptic integral, see [17]:

$$\sum_{n=0}^{\infty} C_n^2 t^2 = \frac{1}{4t^2} \left( -1 + \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - 8t \cos \varphi + 16t^2} d\varphi \right).$$



**Fig. 10.** Left: this meander corresponds to the word  $lr$ ; a word corresponding to a meander, if it is not  $lr$  itself, cannot contain this word as its factor. Middle: a *reducible* meander system; the corresponding word  $lu(ldur)dr$  contains a factor which is itself a meander. Right: this meander system is not connected but it does not contain a factor corresponding to a meander or to a meander system. We call the meander systems as the one on the right *irreducible*.

#### 4. Bounds, estimates, and asymptotics

Anthony Phillips writes in [23]:

In December of 1986 I ran into Paul Erdős at a party and told him about my interest in this combinatorial problem. He asked me whether the number  $M_n$  seemed to be increasing exponentially or even factorially with  $n$ .

Arnold asked us same question. The bound  $M_n \leq C_n^2$  makes the answer evident. It is generally accepted that the numbers  $M_n$  have the asymptotics of the form

$$M_n \sim \text{Const} \cdot A^n n^{-\alpha}; \tag{1}$$

nobody has ever tried to introduce logarithmic terms. The above bounds imply  $4 \leq A \leq 16$ .

According to [23], Larry Shepp proposed the following simple idea. Consider the words of length  $2n$  over the alphabet  $\{l, u, d, r\}$  and forbid the factors  $lr$  (see Fig. 10, left). Then we get a simple rational language accepted by an automaton with three states. This construction gives us the estimate

$$A \leq (2 + \sqrt{3})^2 = 13.92820323 \dots$$

What else would we like to forbid?

It is natural to consider only those meander systems that do not contain a proper factor which is itself a meander system, see Fig. 10. This idea was explored in [17].<sup>3</sup> Numerically, the gain is not very significant. But the bound itself is beautiful:

**Proposition 4.1** (*A Bound for the Growth Rate*). *The growth rate  $A$  admits the following bound:*

$$A \leq \left( \frac{\pi}{4 - \pi} \right)^2 = 13.39408018 \dots$$

The method itself deserves a closer attention. There is a well known and widely used techniques of getting algebraic generating functions from context free languages describing the objects we want to count. In French, context free languages are called «langages algébriques»; we have used, to obtain the above bound, a «langage analytique»: it contained infinitely many production rules. In the final resort, we wrote a *generating function for the production rules* and thus collected them into a single object. The details are rather long to be exposed here; the reader may address the original paper [17].

<sup>3</sup> I saw somewhere that the same bound was also found by A.B.Lehman. He did not publish his result, and I cannot now find the source of this information.



This line of thought was pushed further in [1]. By considering various languages describing subsets of meanders and systems containing meanders, the authors obtained the following bounds:

$$11.380 \leq A \leq 12.901.$$

The numerical estimate of this parameter is  $A = 12.26287 \dots$  (see [13]). The nature of the constant  $A$  is unknown.

Now we come to the most unusual part of the story. Physicists know, and nowadays specialists in enumerative combinatorics know too, that the most important parameter in (1) is the *critical exponent*  $\alpha$ . For physicists, it determines the type of the phase transition; for mathematicians, the type of singularity on the border of the range of convergence of the generating series. In the paragraph that follows we will use a large paraphernalia of theories and terminology from theoretical physics. I do not pretend to understand even a fraction of it. Hence, I systematically use quotation marks to underline the fact that there is a universe of knowledge behind each term. What is even more important, the words like “therefore” will also be put into quotation marks: they do not have, or at least may not have the usual mathematical meaning. Ultimately, even physicists themselves admit that the “result” presented here, no matter how remarkable it is, is not a theorem but only a conjecture. However, this conjecture is not just a wild guess; it is supported by a plausible reasoning à la Pólya and by experimental data.

I follow here the paper [7]. The problem of meanders “may be interpreted” as a model of “a pair of two fully packed loops”. Now, a model of one fully packed loop has a “central charge”  $c = -2$ . “Therefore”, the central charge for the model of two fully packed loops and, hence, for the model of meanders, is  $c = -4$ . There are reasons to believe that the model satisfies the “conformal invariance” property. If this is the case then, using the Knizhnik–Polyakov–Zamolodchikov equation [14], one may express the “string susceptibility”  $\gamma_{\text{str}}$  through the central charge:

$$\gamma_{\text{str}} = \frac{c - 1 - \sqrt{(25 - c)(1 - c)}}{12}.$$

Finally, the critical exponent  $\alpha$  is expressed in terms of the string susceptibility as follows:

$$\alpha = 2 - \gamma_{\text{str}}.$$

For  $c = -4$  all this gives

$$\alpha = \frac{29 + \sqrt{145}}{12} = 3.420132882 \dots \quad (2)$$

As one of our colleagues used to say, « cela ne s’invente pas ».

In [13], a technology is developed of extracting numerical values of parameters of asymptotics of integer sequences from a finite number of known terms. In particular, the authors applied their techniques to the asymptotics of closed meanders and found out that the above value (2) did not contradict their numerical results.

## 5. Computing the exact numbers of meanders

Nobody likes algorithms of exponential complexity, and we all understand why. Meanwhile, such algorithms have an important advantage: if, instead of the complexity  $\sim C^n$ , we get  $\sim (C/k)^n$ , this will give an acceleration with rate  $k^n$ . That is, the improved algorithm will be *exponentially more efficient* than the original one. This is what happened with the use of the algorithm designed by Iwan Jensen [12]. All the previous algorithms had the complexity proportional to the number of meanders, that is, roughly,  $12.26^n$ . The algorithm of Jensen is also exponential, but its complexity is proportional to  $2.58^n$ . Thus, the acceleration factor is, once again very roughly,  $4.75^n$ : quite a feat! The computations that used to take months of work on a multi-processor supercomputer, now took only a few minutes on a modest desktop.

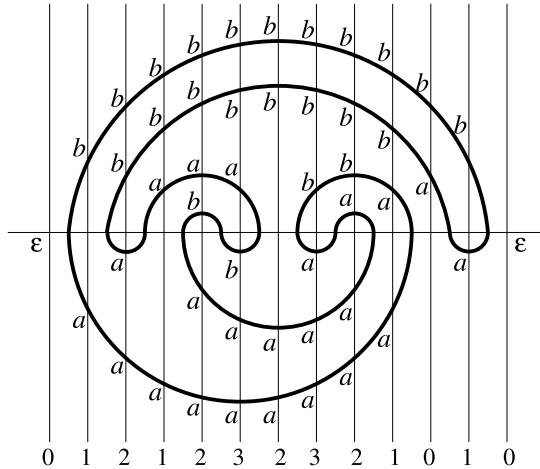
The numbers of closed meanders computed up to now are given in Table 1.

The algorithm designed by Jensen is based on the method of transfer-matrix widely used in statistical physics. Meanders are cut by vertical lines, as is shown in Fig. 11, and every cut is

**Table 1**

Numbers of closed meanders up to  $n = 28$ . Computations by Iwan Jensen (first 24 terms) and Andrew Howroyd; see [29], Entry A005315.

$n$	Nb. of meanders	$n$	Nb. of meanders
1	1	15	602 188 541 928
2	2	16	5 969 806 669 034
3	8	17	59 923 200 729 046
4	42	18	608 188 709 574 124
5	262	19	6 234 277 838 531 806
6	1828	20	64 477 712 119 584 604
7	13 820	21	672 265 814 872 772 972
8	110 954	22	7 060 941 974 458 061 392
9	933 458	23	74 661 728 661 167 809 752
10	8 152 860	24	794 337 831 754 570 367 812
11	73 424 650	25	8 499 066 628 515 413 229 282
12	678 390 116	26	91 412 898 898 828 176 826 244
13	6 405 031 050	27	987 975 910 996 038 555 989 486
14	61 606 881 612	28	10 726 008 363 361 842 734 385 644



**Fig. 11.** A figure from [12], with a slightly changed notation. The sequence of intersection points of the meander with each vertical line is characterized by a Dyck word. The integers at the bottom indicate how many arcs go under the horizontal line.

characterized by a Dyck word (an additional information needed by the algorithm is the number of “letters” below the horizontal line). The matrix  $T$  in question contains the information of the possible “transfers” from a word to the next one which preserve connectedness of the curve. Clearly, the matrix is very sparse. What we need is then to compute the power  $T^{2n+1}$  (here  $2n + 1$  is the number of vertical lines).

The first impression is that the size of the matrix is proportional to the number of Dyck words, that is, to the corresponding Catalan number. It turns out, however, that not all Dyck words may appear in this context. A subtle analysis gives the declared above complexity  $2.58^n$ .

**6. Meander determinant**

I now come to the result which is, to my mind, the most remarkable and also, I would say, the most enigmatic of everything that was done about meanders. It shows that behind these objects there is some hidden and very rich structure.



Fig. 12. Five 3-arc diagrams.

**Prehistory.** In 1989, Witten [32] introduced certain invariants of 3-manifolds which were based on Quantum Field Theory.

In 1991, Lickorish [20] constructed the same invariants using the Jones polynomials; he has also shown that, indeed, they are invariants if the determinant of certain matrix  $\mathcal{M}(q)$  becomes equal to zero when the parameter  $q$  takes the form  $\cos(2\pi/m)$ .

The same year, Ko and Smolinsky [15] proved that this is indeed the case.

Finally, in 1996 Philippe Di Francesco [6] discovered an explicit expression of the determinant  $\det \mathcal{M}(q)$  in terms of the Chebyshev polynomials of the second kind. In his construction he used the fact that the matrix in question is the Gram matrix of a change of bases in the Temperley–Lieb algebra.

We must now introduce the matrix itself.

**The meander matrix.** The matrix  $\mathcal{M}_n(q)$  will be of the size  $C_n \times C_n$ , where  $C_n$  is the  $n$ th Catalan number. Its columns correspond to the arc diagrams of order  $n$ ; its rows, to the same arc diagrams drawn upside-down. When we take a column and a row and glue together the corresponding upper and lower diagrams, we get a meander system. If this meander system has  $k$  components, we put  $q^k$  in the intersection of the corresponding column and row; here  $q$  is a formal parameter. For example, the meander system of Fig. 8 leads to  $q^4$  at the corresponding position in the matrix.

**Example 6.1 (Meander Matrix  $\mathcal{M}_3$ ).** Let us take five 3-arc diagrams, see Fig. 12. They will label the columns of the matrix  $\mathcal{M}_3$ , while the rows will be labeled by the same diagrams upside-down.

As a result, we get the following matrix:

$$\mathcal{M}_3 = \begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}.$$

**Theorem 6.2 (Di Francesco [6]).** The determinant of the matrix  $\mathcal{M}_n$  is equal to

$$\det \mathcal{M}_n = \prod_{k=1}^n U_k(q/2)^{a_{n,k}},$$

where  $U(t)$  are Chebyshev polynomials of the second kind defined by the identity  $U_n(\cos \varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi}$ , while the exponents  $a_{n,k}$  are determined by the following formula:

$$a_{n,k} = \binom{2n}{n-k} - 2 \cdot \binom{2n}{n-k-1} + \binom{2n}{n-k-2} \tag{3}$$

(when  $n-k-1$  and/or  $n-k-2$  become negative, the corresponding binomial coefficient is considered to be equal to zero).

In Example 6.1 we have

$$U_1(q/2) = q, \quad U_2(q/2) = q^2 - 1, \quad U_3(q/2) = q^3 - 2q,$$

and

$$a_{3,1} = \binom{6}{2} - 2 \cdot \binom{6}{1} + \binom{6}{0} = 15 - 2 \cdot 6 + 1 = 4,$$

$$a_{3,2} = \binom{6}{1} - 2 \cdot \binom{6}{0} = 6 - 2 \cdot 1 = 4,$$

$$a_{3,3} = \binom{6}{0} = 1,$$

so that the determinant is

$$\det \mathcal{M}_3 = q^4(q^2 - 1)^4(q^3 - 2q).$$

**Remark 6.3** (*Negative Exponents*). An exponent  $a_{n,k}$  may well become negative; this only means that the corresponding factor is already present in another one, and its degree is greater than needed. Thus, for example, for  $n = 8$  and  $k = 1$  we have

$$a_{8,1} = \binom{16}{7} - 2 \cdot \binom{16}{6} + \binom{16}{5} = 11\,440 - 2 \cdot 8\,008 + 4\,368 = -208.$$

The factor  $q^{-208}$  is here in order to compensate extra degrees of  $q$  coming from powers of the polynomials

$$U_3(q/2) = q^3 - 2q, \quad U_5(q/2) = q^5 - 4q^3 + 3q, \quad U_7(q/2) = q^7 - 6q^5 + 10q^3 - 4q.$$

If we look at the sequence  $\binom{2n}{n-k}$ ,  $k = 1, \dots, n$ , “from far away”, it will resemble the density

of the normal distribution (only the abscissa goes from right to left), while formula (3) for the exponents resembles the second derivative of the density. It is thus not surprising that the second derivative changes its sign somewhere near the standard deviation.

**Remark 6.4** (*Trace of  $\mathcal{M}_n^2$* ). It is easy to see that

$$\text{tr}(\mathcal{M}_n^2) = \sum_{k=1}^n M_{n,k} q^{2k},$$

where  $M_{n,k}$  is the number of meander systems of order  $n$  with  $k$  components. Taking once again [Example 6.1](#) we get

$$\text{tr}(\mathcal{M}_3^2) = 8q^2 + 12q^4 + 5q^6.$$

Unfortunately, nobody has as yet given an explicit formula of this trace.

And why not use this trace in order to compute, say, two more terms in [Table 1](#), that is,  $M_{29}$  and  $M_{30}$ ? Alas, to do that we would need a matrix  $\mathcal{M}_{30}$  of size  $C_{30} \times C_{30}$  where  $C_{30}$  is the 30th Catalan number,

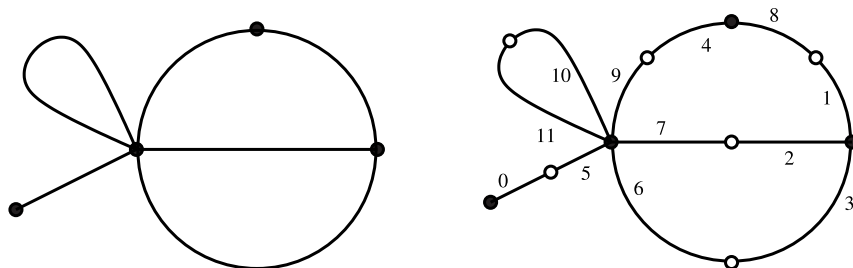
$$C_{30} = 3\,814\,986\,502\,092\,304 \approx 3.8 \cdot 10^{15},$$

while the trace itself is a polynomial of degree 60, maybe cumbersome but certainly manageable.

## 7. Composition of coverings

This section is a bit speculative. I would like to indicate here a direction of research in which meanders (or their generalizations) may eventually play an important role.

The standard way of describing a ramified covering  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are two-dimensional manifolds, is as follows. Let  $B = \{b_1, \dots, b_k\} \subset Y$  be the set of branch points, and let  $y_0$  be a non-branch point. Then we draw loops on  $Y$  attached to  $y_0$ , which generate the fundamental group  $\pi_1(Y \setminus B, y_0)$ ; and then associate to each loop a permutation on the preimage  $X_0 = f^{-1}(y_0)$ . This construction may be found in practically every textbook of algebraic topology. Our remark is that this construction becomes much more manageable and transparent if we consider the preimage not of a single point but of a certain figure drawn on  $Y$  which “involves” in some way the branch points.



**Fig. 13.** Left: an “ordinary” map with six edges. It is obtained as a preimage of a segment via a ramified covering of degree 12. It turns out that this covering is a composition of two coverings, of degree 3 and 4, respectively. Right: we transform this map into a bicolored map with 12 edges by putting a white vertex in the middle of each edge. After that, the 12 edges are labeled.

An example of this construction is given by maps and hypermaps (or, equivalently, maps with bicolored vertices). They are geometric representations of ramified coverings of the sphere  $Y = S^2$  with three ramification points. We color one of these three ramification points in black ( $\bullet$ ), another one, in white ( $\circ$ ), and the third one, say, mark by asterisk ( $*$ ). Then we connect the black and the white points by a segment and lift this segment onto the covering surface  $X$ . What we will see in the latter surface is an embedded bipartite graph, with vertices colored in black and white, and the complement to this graph will be a disjoint collection of topological disks each of which contains a single preimage of the point marked by asterisk. (In pictures, we often do not draw explicitly the asterisks, and when the degrees of white vertices are all equal to 2 we often omit them too, thus obtaining an “ordinary” (i. e., not bicolored) maps.

Other possibilities are also often in use, especially when there are more than three ramification points. One may draw a Jordan curve passing through all ramification points, or else draw a star-tree, with the center at the base point  $y_0$  and the rays going to the points  $b_1, \dots, b_k$ .

**Definition 7.1 (Passport).** The *passport* of a covering of degree  $n$  is the sequence of partitions of  $n$  representing the multiplicities of the preimages of ramification points. Therefore, for a bicolored map the partitions in question represent the degrees of the black vertices, of the white vertices, and of the faces.

**Example 7.2 (Passport).** The map on the right of Fig. 13 has 12 bicolored edges; therefore, the degree of the corresponding covering is 12. Its passport is  $(6^1 3^1 2^1 1^1, 2^6, 6^1 3^1 2^1 1^1)$ . Notice the specific notion of the face degree: it is equal to *half* of the number of edges surrounding the face (otherwise the sum of the degrees would be 24 instead of 12).

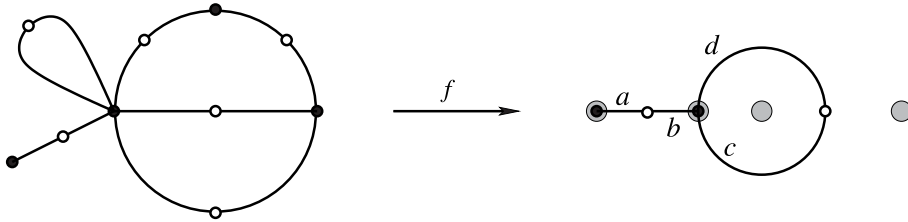
**Exercise 7.3 (Maps with the Same Passport).** There exist, in total, 18 maps with the same passport  $(6^1 3^1 2^1 1^1, 2^6, 6^1 3^1 2^1 1^1)$  as above. Find them all. For four of them, the coverings are decomposable.

**Remark 7.4 (Use Advanced Tools with Care).** If you use GAP in order to find the permutations representing the maps in question, do not forget to impose the transitivity condition, otherwise you will find 20 solutions. If you use the Frobenius formula with characters of the symmetric group in order to find the *number* of the maps in question you will get the answer  $19\frac{1}{2}$ . The reason is, one of the “disconnected maps” has a symmetry of order 2.

The goal of this section is to discuss the situation of the composition of coverings

$$f : X \rightarrow Y, \quad g : Y \rightarrow Z, \quad h = g \circ f : X \rightarrow Z.$$

**Theorem 7.5 (Ritt’s Theorem).** A ramified covering is a composition of coverings of smaller degrees if and only if its monodromy group is imprimitive.



**Fig. 14.** The bicolored map with 12 edges is projected onto a bicolored map with four edges (or, if you ignore the white vertices, on an “ordinary” map with two edges). The black–white edges of the smaller map are labeled by the blocks (4) of the action of the triple of permutations  $\sigma, \alpha, \varphi$ . The four ramification points are indicated by the shaded circles.

A proof, with a detailed discussion, may be found in [34].

Let us look at the map on the left of Fig. 13. It turns out that the covering of degree 12 which produces this map is a composition of two coverings, of degrees 3 and 4, respectively. The reader must agree that this fact does not jump to the eyes. Even the specialists in the theory of *dessins d’enfants* do not find this fact evident.

**Example 7.6 (Composition).** As usual, we represent the map of Fig. 13 by a triple of permutations  $\sigma, \alpha, \varphi$  acting on the bicolored edges:  $\sigma$  rotates the edges around black vertices,  $\alpha$  around white ones, and  $\varphi$  around the face centers, all in the positive direction. (The center of the outer face is situated “on the opposite side” of the sphere; therefore, the direction of the rotation of this face may seem to be negative but in fact it is also positive). We may note that  $\sigma\alpha\varphi = 1$ .

The action of these permutations is imprimitive: considering the blocks

$$a = \{0, 4, 8\}, \quad b = \{1, 5, 9\}, \quad c = \{2, 6, 10\}, \quad d = \{3, 7, 11\} \tag{4}$$

we have

$$\sigma = (0)(1, 2, 3)(4, 8)(5, 6, 7, 9, 10, 11) \rightsquigarrow (a)(b, c, d)(a, a)(b, c, d, b, c, d) \rightarrow (a)(b, c, d),$$

$$\alpha = (0, 5)(1, 8)(2, 7)(3, 6)(4, 9)(10, 11) \rightsquigarrow (a, b)(b, a)(c, d)(d, c)(a, b)(c, d) \rightarrow (a, b)(c, d),$$

$$\varphi = (0, 11, 9, 8, 3, 5)(1, 4, 7)(2, 6)(10) \rightsquigarrow (a, d, b, a, d, b)(b, a, d)(c, c)(c) \rightarrow (a, d, b)(c).$$

The map corresponding to the triple of permutations

$$\bar{\sigma} = (a)(b, c, d), \quad \bar{\alpha} = (a, b)(c, d), \quad \bar{\varphi} = (a, d, b)(c)$$

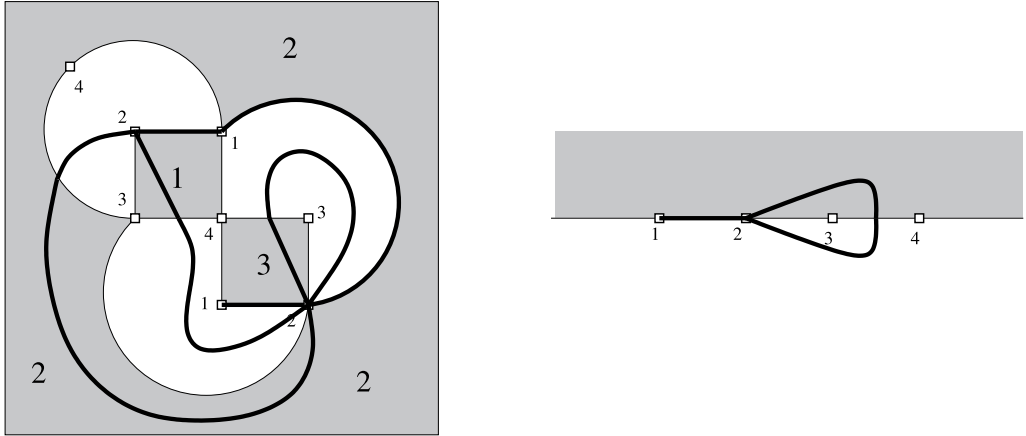
is shown on the right of Fig. 14. There are four ramification points situated on this map: they are marked by shaded circles. For example, the black vertex of degree 3, “described” by the cycle  $(b, c, d)$ , is lifted as  $(b, c, d)(b, c, d, b, c, d)$ , with branching multiplicities  $2^1 1^1$ .

Let us cut the sphere of the smaller map into two parts by the horizontal line passing through the ramification points, and consider the following branching permutations of degree 3 over these points:

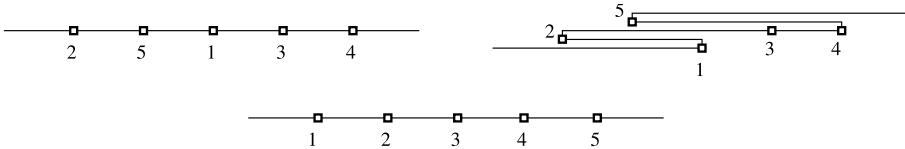
$$s_1 = (1, 2), \quad s_2 = (2, 3), \quad s_3 = (1, 2), \quad s_4 = (1, 3) \quad (\text{notice that } s_1 s_2 s_3 s_4 = 1).$$

The result is shown in Fig. 15. We see three shaded areas and three white areas; in each of them, is repeated the image of the shaded and of the white areas of the smaller map. The resulting *dessin* is isomorphic to the initial map of Fig. 13 (we removed white vertices in order to make the picture less charged).

The reader may agree that the task of decomposing a map with only six edges turns out to be far from trivial, and even when a solution is explicitly presented it is not that easy to understand it. Here is a short and certainly incomplete list of difficulties we may encounter in a more general situation:



**Fig. 15.** The figure drawn in the shaded area on the right is repeated three time in the three shaded quadrilaterals on the left. The same procedure is used for the white areas. Can you recognize, on the left, the initial map of Fig. 13?



**Fig. 16.** The five ramification points lie on a line, but their order is not the one we would like to have. We then imagine the straight line as a broken one, and “mentally” stretch it to obtain a good order.

1. The picture drawn on the intermediate level  $Y$  may be much more complicated than a simple two-edge map on the right of Fig. 15.
2. The ramification points may be many, and they may lie not on a straight line but be scattered about the elements of the map.
3. We may try to draw a Jordan curve passing through the ramification points, but the order of the points on the line may not be the same as the order of permutations representing the covering  $f : X \rightarrow Y$ .

A possible method to cope with the last difficulty is presented in Figs. 16 and 17. We see a meander to appear as a technical tool to represent edges of a map we would like to lift on a covering surface.

Now let us pile on the top of all this one more difficulty we may run into:

4. And what if the surface  $Y$  is not a sphere but a surface of genus  $g \geq 1$ ? In this case a single non self-intersecting curve will not subdivide our surface into two simply connected regions.

All the above discussions lead us to the following rather ambitious project:

*We all know what a giant step forward in the study of maps and hypermaps was their representation by permutations. What we need now is a way, a procedure, a model to represent a pair of maps coexisting and cohabiting on the same surface. From this point of view, meanders represent the simplest possible prototype, when both maps are circles on the sphere.*

### 8. Some recent developments

No end of the story yet in view.



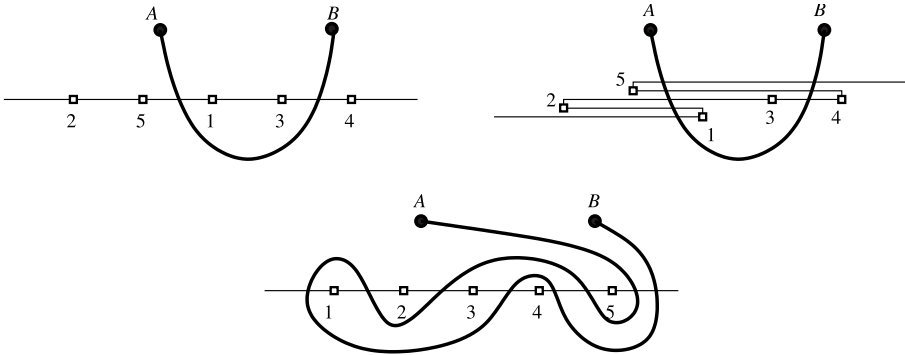


Fig. 17. After the stretching procedure shown in Fig. 16, a simple curve may be transformed into something resembling a meander.

We will not try to represent here the current state of study in the broader field of meandric research. In particular, we will not touch upon applications of meanders in biology and physics, and even on current computational advances (like, for example, [10] or [4]). For all that, one would need a much more extensive overview. We limit ourselves to presenting a few results of only one recent publication.

In 2020, appeared a seminal 80-page paper by Delecroix, Goujard, Zograf and Zorich [5], full of new ideas, new enumerative results, and new relations to various mathematical theories. It goes without saying that it is impossible to give a short summary of the results obtained and techniques used in this paper. This section, and the next one, represent a sort of a trailer, a teaser of a forthcoming thriller.

**Definition 8.1** (*Minimal Arcs and a Rainbow Arc*). An arc is called *minimal* if it connects two neighboring points. An arc is called a *rainbow* if it connects the first point with the last one.

**Example 8.2.** The meander on the left of Fig. 2 has eight minimal arcs and no rainbow arc. The meander of Fig. 11 has six minimal arcs and a rainbow arc.

Concerning the enumeration of meanders on the plane, there are three main novelties in [5] as compared to the usual approach:

- They introduce the new parameter, the number of minimal arcs, and count the meanders with a given number  $p$  of minimal arcs.
- They count not the meanders with a given number  $2n$  of crossings but the meanders with at most  $2n$  crossings.
- And, finally, they count separately meanders with and without the rainbow arc.

Then, the following theorem holds:

**Theorem 8.3** (*Asymptotic Enumeration of Meanders, [5]*). For any fixed  $p$ , the numbers  $M_p^+(n)$  and  $M_p^-(n)$  of closed meanders with  $p$  minimal arcs, with at most  $2n$  crossings, and with or without a rainbow arc, respectively, have the following asymptotics as  $n \rightarrow \infty$ :

$$M_p^+(n) = C_p^+(n) \cdot n^{2p-4} + o(n^{2p-4}) \quad \text{and} \quad M_p^-(n) = C_p^-(n) \cdot n^{2p-5} + o(n^{2p-5}),$$

where

$$C_p^+(n) = \frac{1}{(p-4)p!(p-3)!} \left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2,$$

$$C_p^-(n) = \frac{2}{(2p-5)p!(p-4)!} \left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}.$$

For  $p \gg 1$  one also has the asymptotics

$$C_p^+(n) \sim \frac{\pi^2}{256} \left(\frac{32e^2}{\pi^2 p^2}\right)^p \quad \text{and} \quad C_p^-(n) \sim \frac{\pi^2 e^2}{128p} \left(\frac{32e^2}{\pi^2 p^2}\right)^{p-1}.$$

We see, in particular, that the meanders with a rainbow arc outweigh those without such an arc. The result obviously concerns *plane* meanders. If we join the opposite ends of the line, transforming it into the circle on the sphere, then the rainbow arc becomes a minimal one. The paper also contains enumerative results concerning the *spheric* meanders, and the meanders corresponding to a given topological type of the dual graph, and so on.

We may admire the remarkably explicit nature of the above formulas. What is even more remarkable is the technique of their proof: it uses the *Masur–Veech volumes of the moduli spaces of meromorphic quadratic differentials*. The words are pronounced; now those who have enough courage are invited to go further on this path.

The next subject is even more exciting: more than a half of the paper is devoted to relations of meanders to the *square-tiled surfaces*.

### 9. Square-tiled surfaces, or the theory of origami

Every course of algebraic topology will explain to us that ramified coverings  $f : X \rightarrow Y$  of a surface  $Y$  with the set of ramification points  $B \subset Y$  are classified according to subgroups  $G \leq \pi_1(Y \setminus B)$  where  $\pi_1(Y \setminus B)$  is the fundamental group of the corresponding space (see, for example, Section 1.2 of [19], or Chapters VII and VIII of [31], or else [34]). Coverings of finite degree correspond, in this construction, to subgroups of finite index. We also know that the fundamental group of the sphere with three punctures is  $\pi_1(S^2 \setminus \{\bullet, \circ, *\}) \cong F_2$  where  $F_2$  is the free group with two generators.

Now it is an appropriate moment to make a pause and to ask ourselves: do there exist, beside the thrice-punctured sphere, other surfaces with the same fundamental group  $F_2$ ? Certainly, yes! And the simplest one of them, the closest to the objects we study, is the once-punctured torus. Indeed, the fundamental group of a non-punctured torus is

$$\pi_1(T^2) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle,$$

where  $a$  and  $b$  are a parallel and a meridian on the torus, such that  $aba^{-1}b^{-1}$  represents a contractible cycle. Now, making a puncture destroys the relation  $aba^{-1}b^{-1} = 1$ : the cycle is no longer contractible. What remains is just a pair of generators without any relations between them.

We come to the following conclusion: there is a bijection between coverings of the sphere with three ramification points, and coverings of the torus with one ramification point. Using an altiloquent we may say that there is an equivalence of the categories: that of hypermaps on the one hand, and that of coverings of the torus with one ramification point on the other.

Coverings of the torus also admit geometric representation. We may consider the torus as a square, and put the ramification point at the intersection of the parallel and the meridian. Since there is no ramification points inside the square, the preimage of the square is a collection of squares: the covering surface becomes paved by “small” squares. This is why the surfaces in question are called *square-tiled surfaces*, or *square-ruled surfaces* (as the square-ruled paper), or even *origami*. Unfortunately, I am unable to explain in a concise way how meanders come into this subject.

Notice that the origami surface is never planar: its genus cannot be less than that of the covered surface. Nevertheless, it is possible to transfer the results of one construction to the other. The theory of maps is a vast and well studied domain: enumeration, symmetry, bijections, random maps, and so on, all of them must have their counterparts in the theory of square-tiled surfaces.

From the point of view of *dessins d’enfants* the situation is more complicated. The Riemann complex sphere has only one complex structure. If we rigidify the sphere by putting the point  $\bullet$  to 0,

the point  $\circ$  to 1, and the point  $*$  to  $\infty$ , the preimages of the segment  $[0, 1]$  become themselves rigid: they are isolated points in the corresponding moduli space. On the other hand, the complex torus has infinitely many different complex structures: they form an orbifold of the complex dimension 1. Thus, an origami represents a curve of complex dimension 1 inside the corresponding moduli space. What are the properties of *dessins d'enfants* which may be transferred to these curves is *a priori* not clear but certainly very interesting.

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